Growth rate and extinction rate of a reaction diffusion equation with a singular nonlinearity

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Abstract

We prove the growth rate of global solutions of the equation $u_t = \Delta u - u^{-\nu}$ in $\mathbb{R}^n \times (0, \infty)$, $u(x, 0) = u_0 > 0$ in $\mathbb{R}^n$, where $\nu > 0$ is a constant. More precisely for any $0 < u_0 \in C(\mathbb{R}^n)$ satisfying $A_1(1 + |x|^2)^{\alpha_1} \leq u_0 \leq A_2(1 + |x|^2)^{\alpha_2}$ in $\mathbb{R}^n$ for some constants $1/(1 + \nu) \leq \alpha_1 < 1$, $\alpha_2 \geq \alpha_1$ and $A_2 \geq A_1 = (2\alpha_1(1 - \varepsilon)(n + 2\alpha_1 - 2))^{-1/(1+\nu)}$ where $0 < \varepsilon < 1$ is a constant, the global solution u exists and satisfies $A_1(1 + |x|^2 + b_1 t)^{\alpha_1} \leq u(x, t) \leq A_2(1 + |x|^2 + b_2 t)^{\alpha_2}$ in $\mathbb{R}^n \times (0, \infty)$ where $b_1 = 2(n + 2\alpha_1 - 2)\varepsilon$ and $b_2 = 2n$ if $0 < \alpha_2 \leq 1$ and $b_2 = 2(n + 2\alpha_2 - 2)$ if $\alpha_2 > 1$. When $0 < u_0 \leq A(T_1 + |x|^2)^{1/(1+\nu)}$ in $\mathbb{R}^n$ for some constant $0 < A < ((1 + \nu)/2n)^{1/(1+\nu)}$, we prove that $u(x, t) \leq A(b(T - t) + |x|^2)^{1/(1+\nu)}$ in $\mathbb{R}^n \times (0, T)$ for some constants $b > 0$ and $T > 0$. Hence the solution extincts at the origin at time T. We also find various other conditions for the solution to extinct in a finite time and obtain the corresponding decay rate of the solution near the extinction time.

Key words: Growth rate, extinction rate, reaction diffusion equation, singular nonlinearity

Mathematics Subject Classification: Primary 35B40 Secondary 35B05, 35K50, 35K20

0 Introduction

In this paper we will study the growth rate of global solutions and behaviour near extinction time of the solution of the following Cauchy problem

$$
\begin{cases}
u_t = \Delta u - u^{-\nu} & \text{in } \mathbb{R}^n \times (0, \infty) \\
u(x, 0) = u_0 & \text{in } \mathbb{R}^n
\end{cases}
\tag{0.1}
$$

for any $n \geq 1$ where $\nu > 0$ is a constant. Equation (0.1) arises in many physical models. When $\nu = 2$, (0.1) appears in the modelling of Micro-electromechanical systems (MEMS) which has many applications such as accelerometers for airbag deployment in cars, inkjet printer heads, and the device for the protection of hard disk, etc. Interested readers can read the book, Modeling MEMS and NEMS [15], by J.A. Pelesko and D.H. Berstein for the mathematical modeling and various applications of MEMS devices.

Recently there are a lot of research on (0.1) by N. Ghoussoub, Y. Guo, Z. Pan and M.J. Ward [3], [4], [5], N.I. Kavallaris, T. Miyasita and T. Suzuki [11], F. Lin and Y. Yang [13], L. Ma, Z. Guo and J. Wei [6], [7], [14], etc. on the various properties of the solutions of the equation. Note that the stationary solution of (0.1) is

$$\Delta u = u^{-\nu} \quad \text{in } \mathbb{R}^n \tag{0.2}$$

is studied extensively in [6]. An equation similar to (0.2) arising from the motion of thin films of viscous fluid is also studied by H. Jiang and W.M. Ni in [10].

In [6] Z. Guo and J. Wei constructed solutions of (0.1) using a fixed point argument. Then by carefully studying the properties of the solutions of (0.2) Z. Guo and J. Wei [6] proved that if $\nu > 0$, $n \geq 3$,

$$u_0 \in C_{LB}(\mathbb{R}^n) = \{\phi \in C(\mathbb{R}^n) : \inf_{\mathbb{R}^n} \phi > 0 \text{ and } \exists \alpha \leq 0, A > 0 \text{ and } M > 0 \text{ such that}$$

$$\phi(x) \leq A|x|^\alpha \quad \forall |x| \geq M\}$$

and

$$u_0 \geq \gamma \left[\frac{2}{\nu+1} \left(n - 2 + \frac{2}{\nu+1} \right) \right]^{-\frac{1}{1+\nu}} |x|^{\frac{2}{1+\nu}}$$

for some constant $\gamma > 1$, then (0.1) has a unique global solution u satisfying

$$u(x,t) \geq \gamma \left[\frac{2}{\nu+1} \left(n - 2 + \frac{2}{\nu+1} \right) \right]^{-\frac{1}{1+\nu}} |x|^{\frac{2}{1+\nu}}$$

and

$$u(x,t) \geq (\nu+1)^{\frac{1}{1+\nu}} (\gamma^{\nu+1} - 1)^{\frac{1}{1+\nu}} t^{\frac{1}{1+\nu}}.$$

They also used a contradiction argument to prove the finite extinction property of the solution of (0.1) when $n \geq 3$ and the initial value u_0 is bounded above by the supersolution of (0.2).

In this paper by approximating the solution of (0.1) by solutions of (0.1) in bounded domains we prove that for any $n \geq 1$ if the initial value u_0 satisfies

$$A_1(1 + |x|^2)^{\alpha_1} \leq u_0 \leq A_2(1 + |x|^2)^{\alpha_2} \quad \text{in } \mathbb{R}^n \tag{0.3}$$

for some constants $1/(1+\nu) \leq \alpha_1 < 1$, $\alpha_2 \geq \alpha_2$, $\nu > 0$, $A_2 \geq A_1$ where

$$A_1 = [2\alpha_1(1-\varepsilon)(n + 2\alpha_1 - 2)]^{-\frac{1}{1+\nu}} \tag{0.4}$$

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for some constant $0 < \varepsilon < 1$, then

$$A_1(1 + |x|^2 + b_1 t)^{\alpha_1} \le u(x,t) \le A_2(1 + |x|^2 + b_2 t)^{\alpha_2} \qquad (0.5)$$

in $\mathbb{R}^n \times (0, \infty)$ where

$$b_1 = 2(n + 2\alpha_1 - 2)\varepsilon \qquad (0.6)$$

and

$$b_2 = \begin{cases} 2n & \text{if } 1/(1+\nu) \le \alpha_2 \le 1 \\ 2(n + 2\alpha_2 - 2) & \text{if } \alpha_2 > 1. \end{cases} \qquad (0.7)$$

Finite time extinction of solutions of (0.1) when the initial value u_0 is bounded above by the supersolution of (0.2) has been proved in [6]. However there is no estimate for the extinction rate and extinction time of the solution in [6]. In this paper we will prove the extinction rate and find an explicit upper bound for the extinction time of the solutions of (0.1) when $u_0 \in C(\mathbb{R}^n)$, $n \ge 1$, satisfies either

$$0 < u_0 \le A(T_1 + |x|^2)^{\frac{1}{1+\nu}} \quad \forall x \in \mathbb{R}^n \qquad (0.8)$$

for some constants

$$T_1 > 0 \quad \text{and} \quad 0 < A < \left(\frac{1+\nu}{2n} \right)^{\frac{1}{1+\nu}}, \qquad (0.9)$$

or

$$0 < u_0 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

We also find various other conditions on the initial value for the solution of (0.1) to extinct in a finite time and prove the corresponding extinction rate.

The plan of the paper is as follows. In section 1 we will construct the solution of (0.1) by approximating the solution of (0.1) by solutions of (0.1) in bounded domains. We will construct explicit supersolutions and subsolutions of (0.1) and use them to prove that the solutions in bounded domain have the bounds we want. Then by an approximation argument the global solution will have the same upper and lower bounds. In section 2 we will prove the extinction rate and extinction time of solutions of (0.1) under various conditions on the initial value.

We start will some definitions. For any $R > 0$, $T > 0$, let $B_R = \{x \in \mathbb{R}^n : |x| < R\}$, $Q_R^T = B_R \times (0,T)$ and $Q_R = B_R \times (0,\infty)$. For any $f \in C(\partial B_R \times (0,T)) \cap L^\infty(B_R \times (0,T))$ we say that $0 < u \in C([0,T]; L^1(B_R)) \cap L^\infty(Q_R^T)$ is a continuous weak solution (subsolution, supersolution respectively) of

$$\begin{cases} u_t = \Delta u - u^{-\nu} & \text{in } Q_R^T \\ u = f & \text{on } \partial B_R \times (0,T) \\ u(x,0) = u_0 & \text{in } B_R \end{cases} \qquad (0.10)$$

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if u satisfies the integral identity

$$\iint_{B_R \times [t_1, t_2]} (u\eta_t + u\Delta\eta - u^{-\nu}\eta)\, dx\, dt = \int_{t_1}^{t_2} \int_{\partial B_R} f\frac{\partial \eta}{\partial N}\, d\sigma\, dt + \int_{B_R} u\eta\, dx\Big|_{t=t_1}^{t=t_2}$$

($\geq, \leq$ respectively) for any $0 < t_1 < t_2 < T$, $\eta \in C^\infty(\overline{Q_R^T})$ such that $\eta = 0$ on $\partial B_R \times [0, T]$ where $\partial/\partial N$ is the exterior normal derivative on $\partial B_R \times (0, T)$.

We say that u is a solution (subsolution, supersolution respectively) of (0.10) if $0 < u \in C^{2,1}(Q_R^T) \cap C(\overline{B}_R \times (0, T))$ satisfies ($\leq, \geq$ respectively)

$$u_t = \Delta u - u^{-\nu} \tag{0.11}$$

in Q_R^T in the classical sense with $u = f$ ($\leq f, \geq f$ respectively) on $\partial B_R \times (0, T)$ and

$$\lim_{t \to 0} \int_{B_R} |u - u_0|\, dx = 0.$$

We say that u is a solution (subsolution, supersolution respectively) of (0.1) in $\mathbb{R}^n \times (0, T)$ if $u \in C^{2,1}(\mathbb{R}^n \times (0, T))$ satisfies (0.11) ($\leq, \geq$ respectively) in $\mathbb{R}^n \times (0, T)$ in the classical sense, $u(\cdot, t)$ converges to u_0 in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0$, and for any $0 < \delta < T$ there exist constants $C_1 = C_1(\delta, T) > 0$, $C_2 = C_2(\delta, T) > 0$, such that

$$C_1(1 + |x|^2)^{-\frac{1}{1+\nu}} \leq u_2(x, t) \leq C_2 e^{C_2|x|^2} \quad \text{in } \mathbb{R}^n \times (0, T - \delta).$$

We say that u is a solution (subsolution, supersolution respectively) of (0.1) in $\mathbb{R}^n \times (0, \infty)$ if $u \in C^{2,1}(\mathbb{R}^n \times (0, \infty))$ is a solution of (0.1) in $\mathbb{R}^n \times (0, T)$ (subsolution, supersolution respectively) for any $T > 0$.

Let $G_R(x, y, t)$, $x, y \in B_R$, $t > 0$, be the Dirichlet Green function of the heat equation in Q_R. That is for any $y \in B_R$,

$$\begin{cases} \partial_t G_R = \Delta_x G_R & \text{in } Q_R \\ G_R(x, y, t) = 0 & \forall x \in \partial B_R, t > 0 \\ \lim_{t \to 0} G_R(x, y, t) = \delta_y \end{cases}$$

where δ_y is the delta mass at y. By the maximum principle,

$$0 \leq G_R(x, y, t) \leq \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-|x-y|^2/4t} \quad \forall x, y \in B_R, t > 0.$$

For any $K \subset R^n \times (0, \infty)$, $0 < \beta < 1$, let

$$C^{2,1}(K) = \{f : f, f_t, f_{x_i}, f_{x_i, x_j} \in C(K) \quad \forall i, j = 1, 2, \ldots, n\}$$

and let $C^{2+\beta, 1+(\beta/2)}(K)$ denote the class of all functions $f \in C^{2,1}(K)$ such that

$$\begin{cases} |f_{x_i, x_j}(x_1', t_1') - f_{x_i, x_j}(x_2', t_2')| \leq C(|x_1' - x_2'|^\beta + |t_2' - t_1'|^{\beta/2})\} & \forall (x_1', t_1'), (x_2', t_2') \in K \\ |f_t(x_1', t_1') - f_t(x_2', t_2')| \leq C(|x_1' - x_2'|^\beta + |t_2' - t_1'|^{\beta/2})\} & \forall (x_1', t_1'), (x_2', t_2') \in K \end{cases}$$

holds for some constant $C > 0$ and any $i, j = 1, 2, \cdots, n$.

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1 Existence and growth rate of global solutions

In this section we will construct explicit supersolutions and subsolutions of (0.1). We will also give a different proof of the existence of global solutions of (0.1) and prove the growth rate estimates of the global solutions of (0.1).

We first observe that by an argument similar to the proof of Theorem 1.1 of [9] (cf. Lemma 2.3 of [1]) we have the following theorem.

Lemma 1.1. *Let $u_{0,1}, u_{0,2} \in L^1(B_R)$ be such that $0 \le u_{0,1} \le u_{0,2}$ a.e. on B_R. Let $f_1, f_2 \in C(\partial B_R \times (0,T)) \cap L^\infty(\partial B_R \times (0,T))$ be such that $0 < f_1 \le f_2$ on $\partial B_R \times (0,T)$. Let u_1, u_2, be continuous weak subsolution and supersolution of (0.10) with initial value $u_0 = u_{0,1}, u_{0,2}$, and $f = f_1, f_2$ respectively. Suppose there exists a constant $\delta > 0$ such that $u_1 \ge \delta$ and $u_2 \ge \delta$ on Q_R^T. Then*

$$u_1(x,t) \le u_2(x,t) \quad \forall x \in \overline{B}_R, 0 \le t < T.$$

Lemma 1.2. *Let $\nu > 0$, $u_0 \in C(\overline{B}_R)$ and $f \in C(\partial B_R \times (0,T_1)) \cap L^\infty(\partial B_R \times (0,T))$. Suppose $\delta = \min(\min_{\overline{B}_R} u_0, \inf_{\partial B_R \times (0,T_1)} f) > 0$. Then there exists a constant $0 < T < T_1$ such that (0.10) has a unique solution u which satisfies*

$$\frac{\delta}{2} \le u \le w_R \quad \forall x \in B_R, 0 < t < T \tag{1.1}$$

and

$$u(x,t) = \int_{B_R} G_R(x,y,t) u_0(y)\, dy - \int_0^t \int_{\partial B_R} \frac{\partial G_R}{\partial N_y}(x,y,t-s) f(y,s)\, d\sigma(y)\, ds$$
$$- \int_0^t \int_{B_R} G_R(x,y,t-s) u(y,s)^{-\nu}\, dy\, ds \quad \forall x \in B_R, 0 < t < T \tag{1.2}$$

where $\partial/\partial N_y$ is the exterior normal derivative on $\partial B_R \times (0,T)$ and w_R is the solution of problem

$$\begin{cases} w_t = \Delta w & \text{in } Q_R \\ w = f & \text{on } \partial B_R \times (0,\infty) \\ w(x,0) = u_0 & \text{in } B_R \end{cases} \tag{1.3}$$

Proof: Since uniqueness of solution of (0.10) follows directly by Lemma 1.1, we only need to prove existence of solution of (0.10). Let $u_1(x,t) \equiv \delta$. For any $k \ge 2$, let

$$u_k(x,t) = \int_{B_R} G_R(x,y,t) u_0(y)\, dy - \int_0^t \int_{\partial B_R} \frac{\partial G_R}{\partial N_y}(x,y,t-s) f(y,s)\, d\sigma(y)\, ds$$
$$- \int_0^t \int_{B_R} G_R(x,y,t-s) u_{k-1}(y,s)^{-\nu}\, dy\, ds. \tag{1.4}$$

Note that by standard parabolic theory [12] the solution of (1.3) is given by

$$w_R(x,t) = \int_{B_R} G_R(x,y,t)u_0(y)\,dy - \int_0^t \int_{\partial B_R} \frac{\partial G_R}{\partial N_y}(x,y,t-s)f(y,s)\,d\sigma(y)\,ds. \quad (1.5)$$

Then by (1.4) and (1.5),

$$u_k(x,t) \le w_R(x,t) \quad \forall |x| \le R, 0 < t < T_1, k \ge 2. \quad (1.6)$$

By the maximum principle,

$$w_R \ge \delta \quad \text{in } Q_R^{T_1}.$$

Let

$$T = \min(T_1/2, (\delta/2)^{1+\nu}). \quad (1.7)$$

Then by (1.4) $\forall (x,t) \in \overline{Q_R^T}$,

$$u_2(x,t) \ge \delta - (\delta/2)^{-\nu} \int_0^t \int_{B_R} G(x,y,t-s)\,dy\,ds \ge \delta - (\delta/2)^{-\nu}t \ge \delta/2.$$

Suppose $u_k(x,t) \ge \delta/2$ for any $(x,t) \in \overline{Q_R^T}$ for some $k \ge 2$. Then by (1.4) for any $(x,t) \in \overline{Q_R^T}$,

$$u_{k+1}(x,t) \ge \delta - (\delta/2)^{-\nu} \int_0^t \int_{B_R} G(x,y,t-s)\,dy\,ds \ge \delta - (\delta/2)^{-\nu}t \ge \delta/2.$$

Hence by induction

$$u_k(x,t) \ge \delta/2 \quad \forall (x,t) \in \overline{Q_R^T}, k \in \mathbb{Z}^+. \quad (1.8)$$

We now divide the proof into two cases.
Case 1: $u_0 \in C^\infty(\overline{B_R})$ and $f \in C^\infty(\partial B_R \times [0,T_1])$ with $u(x,0) = f(x,0)$ on ∂B_R.

By (1.4), (1.6), (1.8) and the parabolic Schauder estimates [12], the sequence $\{u_k\}_{k=1}^\infty$ are uniformly Holder continuous on $\overline{Q_R^T}$. Then by the parabolic Schauder estimates ([2],[12]) the sequence $\{u_k\}_{k=1}^\infty$ are uniformly bounded in $C^{2+\beta,1+(\beta/2)}(K)$ for any compact subset $K \subset \overline{B_R} \times (0,T]$ where $0 < \beta < 1$ is some constant. By the Ascoli theorem and a diagonalization argument $\{u_k\}_{k=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself which converges uniformly in $C^{2+\beta,1+(\beta/2)}(K)$ to some function $u \in C(\overline{Q_R^T}) \cap C^{2+\beta,1+(\beta/2)}(K)$ for any compact subset $K \subset \overline{B_R} \times (0,T]$ as $k \to \infty$. Letting $k \to \infty$ in (1.4), (1.6) and (1.8), we get (1.1) and (1.2). By (1.4) u_k satisfies

$$\begin{cases} u_{k,t} = \Delta u_k - u_{k-1}^{-\nu} & \text{in } Q_R^T \\ u_k = f & \text{on } \partial B_R \times (0,T) \\ u_k(x,0) = u_0 & \text{on } B_R \end{cases} \quad (1.9)$$

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Letting $k \to \infty$ in (1.9), u satisfies (0.10).

Case 2: $u_0 \in C(\overline{B}_R)$ and $f \in C(\partial B_R \times [0, T_1])$.

Let T be given by (1.7) and let $\widetilde{u}_1 \equiv \delta$. For any $k \geq 2$, let

$$\widetilde{u}_k(x,t) = \delta \int_{B_R} G_R(x,y,t)\,dy - \delta \int_0^t \int_{\partial B_R} \frac{\partial G_R}{\partial N_y}(x,y,t-s)\,d\sigma(y)\,ds$$

$$- \int_0^t \int_{B_R} G_R(x,y,t-s)\widetilde{u}_{k-1}(y,s)^{-\nu}\,dy\,ds$$

$$= \delta - \int_0^t \int_{B_R} G_R(x,y,t-s)\widetilde{u}_{k-1}(y,s)^{-\nu}\,dy\,ds. \tag{1.10}$$

Then

$$\widetilde{u}_k(x,t) \leq \delta \quad \text{in } Q_R^{T_1} \quad \forall k \in \mathbb{Z}^+. \tag{1.11}$$

By the same argument as before

$$\widetilde{u}_k(x,t) \geq \delta/2 \quad \text{in } Q_R^T \quad \forall k \in \mathbb{Z}^+. \tag{1.12}$$

Hence by case 1 $\widetilde{u}_k$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly in $C^{2+\beta, 1+(\beta/2)}(K)$ to some function $\widetilde{u} \in C(\overline{Q_R^T}) \cap C^{2+\beta, 1+(\beta/2)}(K)$ for any compact subset $K \subset \overline{B}_R \times (0, T]$ as $k \to \infty$. Moreover $\widetilde{u}$ is a solution of (0.10) with u_0 and f being replaced by δ and satisfies

$$\delta/2 \leq \widetilde{u}(x,t) \leq \delta \quad \text{in } Q_R^T. \tag{1.13}$$

Now since $u_1 \equiv \widetilde{u}_1$ in Q_R^T, by (1.4) and (1.10), $\widetilde{u}_2 \leq u_2$ in Q_R^T. Suppose

$$\widetilde{u}_k \leq u_k \quad \text{in } Q_R^T. \tag{1.14}$$

for some $k \geq 2$. Then by (1.4), (1.10) and (1.14) we get that (1.14) holds with k replaced by $k+1$. Hence by induction (1.14) holds for all $k \in \mathbb{Z}^+$. Since $\widetilde{u}_k$ converge uniformly to $\widetilde{u}$ on $\overline{Q_R^T}$ as $k \to \infty$, by (1.6) and (1.14) and the parabolic Schauder estimates ([2],[12]) the sequence $\{u_k\}_{k=1}^\infty$ are uniformly bounded in $C^{2+\beta, 1+(\beta/2)}(K)$ for any compact subset $K \subset B_R \times (0, T]$ where $0 < \beta < 1$ is some constant. By the Ascoli theorem and a diagonalization argument $\{u_k\}_{k=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself which converges uniformly in $C^{2+\beta, 1+(\beta/2)}(K)$ to some function $u \in C(Q_R^T) \cap C^{2+\beta, 1+(\beta/2)}(K)$ for any compact subset $K \subset B_R \times (0, T]$ as $k \to \infty$.

By (1.4) u_k satisfies (1.9). Letting $k \to \infty$ in (1.4) and (1.9) u satisfies (1.2) and (0.11) in Q_R^T. Then by (1.2) $u \in C(\overline{B}_R \times (0, T))$. Since the last two terms of (1.2) vanish as $t \to 0$ and the first term on the right hand side is the solution of (1.3) with $f = 0$ which converges to u_0 in $L^1(B_R)$ as $t \to 0$, u converges to u_0 in $L^1(B_R)$ as $t \to 0$. Hence u is the solution of (0.10). Letting $k \to \infty$ in (1.6) and (1.14), by (1.13) we get (1.1) and the lemma follows. $\square$

Theorem 1.3. *Let $\nu > 0$, $\alpha_2 \geq \alpha_1 \geq 1/(1+\nu)$ and let*

$$A_2 \geq A_1 = \begin{cases} [2\alpha_1(1-\varepsilon)(n+2\alpha_1-2)]^{-\frac{1}{1+\nu}} & \text{if } 1/(1+\nu) \leq \alpha_1 \leq 1 \\ [2\alpha_1(1-\varepsilon)n]^{-\frac{1}{1+\nu}} & \text{if } \alpha_1 > 1 \end{cases} \tag{1.15}$$

for some constant $0 < \varepsilon < 1$. Suppose u_0 satisfies (0.3). Let $f \in C(\partial B_R \times [0,\infty))$ be such that

$$A_1(1+|x|^2+b_1 t)^{\alpha_1} \leq f \leq A_2(1+|x|^2+b_2 t)^{\alpha_2} \qquad \text{on } \partial B_R \times [0,\infty)$$

where

$$b_1 = \begin{cases} 2(n+2\alpha_1-2)\varepsilon & \text{if } 1/(1+\nu) \leq \alpha_1 \leq 1 \\ 2\varepsilon n & \text{if } \alpha_1 > 1 \end{cases} \tag{1.16}$$

and b_2 is given by (0.7). Then there exists a unique solution u of (0.10) in Q_R which satisfies (1.2) and (0.5) in Q_R.

Proof: Let $\psi_i = A_i(1+|x|^2+b_i t)^{\alpha_i}$ for $i = 1,2$. By direct computation, $\forall i = 1,2$,

$$\Delta\psi_i = 2\alpha_i A_i \left\{ n + 2\alpha_i - 2 + 2(1-\alpha_i)\frac{1+b_i t}{1+|x|^2+b_i t} \right\}(1+|x|^2+b_i t)^{\alpha_i-1} \tag{1.17}$$

Since

$$2(1-\alpha_1)\frac{1+b_1 t}{1+|x|^2+b_1 t} \geq \begin{cases} 0 & \text{if } 1/(1+\nu) \leq \alpha_1 \leq 1 \\ 2(1-\alpha_1) & \text{if } \alpha_1 > 1, \end{cases}$$

by (1.17),

$$\Delta\psi_1 \geq \begin{cases} 2\alpha_1 A_1(n+2\alpha_1-2)(1+|x|^2+b_1 t)^{\alpha_1-1} & \text{if } 1/(1+\nu) \leq \alpha_1 \leq 1 \\ 2\alpha_1 n A_1(1+|x|^2+b_1 t)^{\alpha_1-1} & \text{if } \alpha_1 > 1. \end{cases}$$

Hence for $1/(1+\nu) \leq \alpha_1 \leq 1$,

$$\begin{aligned}
&\Delta\psi_1 - \psi_1^{-\nu} - \psi_{1,t} \\
&\geq 2\alpha_1 A_1(n+2\alpha_1-2)(1+|x|^2+b_1 t)^{\alpha_1-1} - A_1^{-\nu}(1+|x|^2+b_1 t)^{-\alpha_1\nu} \\
&\quad - \alpha_1 b_1 A_1(1+|x|^2+b_1 t)^{\alpha_1-1} \\
&\geq A_1^{-\nu}(1+|x|^2+b_1 t)^{-\alpha_1\nu}\{\alpha_1(2(n+2\alpha_1-2)-b_1)A_1^{1+\nu}(1+|x|^2+b_1 t)^{\alpha_1(1+\nu)-1} - 1\} \\
&\geq A_1^{-\nu}(1+|x|^2+b_1 t)^{-\alpha_1\nu}\{\alpha_1(2(n+2\alpha_1-2)-b_1)A_1^{1+\nu} - 1\} \tag{1.18}
\end{aligned}$$

and for $\alpha_1 > 1$,

$$\begin{aligned}
&\Delta\psi_1 - \psi_1^{-\nu} - \psi_{1,t} \\
&\geq 2\alpha_1 n A_1(1+|x|^2+b_1 t)^{\alpha_1-1} - A_1^{-\nu}(1+|x|^2+b_1 t)^{-\alpha_1\nu} \\
&\quad - \alpha_1 b_1 A_1(1+|x|^2+b_1 t)^{\alpha_1-1} \\
&\geq A_1^{-\nu}(1+|x|^2+b_1 t)^{-\alpha_1\nu}\{\alpha_1(2n-b_1)A_1^{1+\nu}(1+|x|^2+b_1 t)^{\alpha_1(1+\nu)-1} - 1\} \\
&\geq A_1^{-\nu}(1+|x|^2+b_1 t)^{-\alpha_1\nu}\{\alpha_1(2n-b_1)A_1^{1+\nu} - 1\}. \tag{1.19}
\end{aligned}$$

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By (1.15) and (1.16) the right hand side of (1.18) and (1.19) is ≥ 0. Hence

$$\Delta\psi_1 - \psi_1^{-\nu} \geq \psi_{1,t} \quad \text{in } Q_R.$$

Since

$$2(1-\alpha_2)\frac{1+b_2 t}{1+|x|^2+b_2 t} \leq \begin{cases} 2(1-\alpha_2) & \text{if } 1/(1+\nu) \leq \alpha_2 \leq 1 \\ 0 & \text{if } \alpha_2 > 1, \end{cases}$$

by (1.17),

$$\Delta\psi_2 \leq \begin{cases} 2\alpha_2 n A_2(1+|x|^2+b_2 t)^{\alpha_2-1} & \text{if } 1/(1+\nu) \leq \alpha_2 \leq 1 \\ 2\alpha_2 A_2(n+2\alpha_1-2)(1+|x|^2+b_2 t)^{\alpha_2-1} & \text{if } \alpha_2 > 1. \end{cases}$$

By (0.7) for $1/(1+\nu) \leq \alpha_2 \leq 1$,

$$\begin{aligned}
&\Delta\psi_2 - \psi_2^{-\nu} - \psi_{2,t} \\
&\leq 2\alpha_2 n A_2(1+|x|^2+b_2 t)^{\alpha_2-1} - A_2^{-\nu}(1+|x|^2+b_2 t)^{-\alpha_2\nu} \\
&\quad - \alpha_2 b_2 A_2(1+|x|^2+b_2 t)^{\alpha_2-1} \\
&\leq A_2^{-\nu}(1+|x|^2+b_2 t)^{-\alpha_2\nu}\{\alpha_2(2n-b_2)A_2^{1+\nu}(1+|x|^2+b_2 t)^{\alpha_2(1+\nu)-1} - 1\} \\
&\leq 0 \quad \text{in } Q_R
\end{aligned}$$

and for $\alpha_1 > 1$,

$$\begin{aligned}
&\Delta\psi_2 - \psi_2^{-\nu} - \psi_{2,t} \\
&\leq 2\alpha_2 A_2(n+2\alpha_2-2)(1+|x|^2+b_2 t)^{\alpha_2-1} - A_2^{-\nu}(1+|x|^2+b_2 t)^{-\alpha_2\nu} \\
&\quad - \alpha_2 b_2 A_2(1+|x|^2+b_2 t)^{\alpha_2-1} \\
&\leq A_2^{-\nu}(1+|x|^2+b_2 t)^{-\alpha_2\nu}\{\alpha_2(2(n+2\alpha_2-2)-b_2)A_2^{1+\nu}(1+|x|^2+b_2 t)^{\alpha_2(1+\nu)-1} - 1\} \\
&\leq 0 \quad \text{in } Q_R.
\end{aligned}$$

By Lemma 1.2 there exists a constant $T > 0$ such that there exists a unique solution u of (0.10) in Q_R^T which satisfies (1.1) and (1.2) in Q_R^T with $\delta = A_1$. Let $T_1 \geq T$ be the maximal existence time of a unique solution of u of (0.10) in $Q_R^{T_1}$. Suppose $T_1 < \infty$. For any $0 < \delta < T_1/2$ since $u \in C(\overline{B}_R \times [T_1/2, T_1-\delta])$ is a classical solution of (0.11), $\min_{\overline{B}_R \times [T_1/2, T_1-\delta]} u > 0$. Hence by (1.2) $\inf_{Q_R^{T_1-\delta}} u > 0$. Since ψ_1 and ψ_2 are subsolution and supersolution of (0.10), by Lemma 1.1 (0.5) holds in $Q_R^{T_1-\delta}$ for any $0 < \delta < T_1/2$. Hence (0.5) holds in $Q_R^{T_1}$.

Then by (0.5) and the parabolic Schauder estimates [12] u can be extended to a continuous function on $\overline{B}_R \times [T_1/2, T_1]$. Then by Lemma 1.2 there exists a constant $T_3 > 0$ such that there exists a solution $\widetilde{u}(x,t)$ of (0.10) in $Q_R^{T_3}$ with $u_0 = u(x, T_1)$ and f being replaced by $f(x, T_1+t)$. We extend u to a function on $\overline{B}_R \times (0, T_1+T_3)$ by setting $u(x,t) = \widetilde{u}(x, T_1+t)$ for any $|x| \leq R$, $T_1 \leq t < T_1+T_3$. Then u is a solution of (0.10) in $Q_R^{T_1+T_3}$. This contradicts the maximality of T_1. Hence $T_1 = \infty$. By the

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previous argument u satisfies (0.5) in Q_R. Let v be given by the right hand side of (1.2). Then v satisfies

$$\begin{cases} v_t = \Delta v - u^{-\nu} & \text{in } Q_R \\ v = f & \text{on } \partial B_R \times (0, \infty) \\ v(x, 0) = u_0 & \text{in } B_R. \end{cases}$$

Hence the function $w = u - v$ satisfies

$$\begin{cases} w_t = \Delta w & \text{in } Q_R \\ w = 0 & \text{on } \partial B_R \times (0, \infty) \\ w(x, 0) = 0 & \text{in } B_R. \end{cases}$$

By the maximum principle $w \equiv 0$ in Q_R. Hence $u = v$ in Q_R. Thus u satisfies (1.2) and the theorem follows. $\qquad \square$

We next recall a comparison result of [16].

Lemma 1.4. *(Lemma 1.3 of [16]) Suppose $\overline{u}$ and $\underline{u}$ are supersolution and subsolution of*

$$\begin{cases} u_t = \Delta u + f(u) & \text{in } \mathbb{R}^n \times (0, T) \\ u(x, 0) = u_0 & \text{in } \mathbb{R}^n \end{cases}$$

for some function $u_0 \in C(\mathbb{R}^n)$ and $f \in C(\mathbb{R})$ such that $\overline{u} - \underline{u} \geq -Be^{\beta|x|^2}$ in $\mathbb{R}^n \times (0, T)$ for some constants $B > 0$ and $\beta > 0$. Suppose $f(\overline{u}) - f(\underline{u}) \geq g(x, t)(\overline{u} - \underline{u})$ for some function $g \in C_{loc}^{\alpha, \alpha/2}(\mathbb{R}^n \times (0, T))$, $0 < \alpha < 1$, and $g(x, t) \leq C(1 + |x|^2)$ on $\mathbb{R}^n \times (0, T)$ for some constant $C > 0$. Then $\overline{u} \geq \underline{u}$ on $\mathbb{R}^n \times (0, T)$.

Theorem 1.5. *Let $\nu > 0$, $1/(1 + \nu) \leq \alpha_1 \leq \alpha_2$, $\alpha_1 < 1$ and $A_2 \geq A_1$ where A_1 is given by (0.4) for some constant $0 < \varepsilon < 1$. Suppose u_0 satisfies (0.3). Then there exists a unique solution u of (0.1) which satisfies*

$$u(x, t) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} u_0(y) \, dy - \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} u(y, s)^{-\nu} \, dy \, ds$$

(1.20)

and (0.5) in $\mathbb{R}^n \times (0, \infty)$ where b_1 and b_2 are given by (0.6) and (0.7).

Proof: By Theorem 1.3 for any $k \in \mathbb{Z}^+$ there exists a unique solution u_k of (0.10) with $f = A_1(1 + |x|^2 + b_1 t)^{\alpha_1}$ in Q_k which satisfies (0.5) and

$$u_k(x, t) = \int_{B_R} G_k(x, y, t) u_0(y) \, dy - \int_0^t \int_{B_k} G_k(x, y, t - s) u_k(y, s)^{-\nu} \, dy \, ds$$

$$- A_1 \int_0^t \int_{\partial B_k} \frac{\partial G_k}{\partial N_y}(x, y, t - s)(1 + |y|^2 + b_1 s)^{\alpha_1} \, d\sigma(y) \, ds. \qquad (1.21)$$

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in Q_k. By (0.5) and the parabolic Schauder estimates [12] the sequence $\{u_k\}_{k=1}^{\infty}$ is uniformly bounded in $C^{2+\beta,1+(\beta/2)}(K)$ for any compact subset $K \subset \mathbb{R}^n \times (0,\infty)$ where $0 < \beta < 1$ is some constant. By the Ascoli theorem and a diagonalization argument $\{u_k\}_{k=1}^{\infty}$ has a subsequence which we may assume without loss of generality to be the sequence itself which converges uniformly in $C^{2+\beta,1+(\beta/2)}(K)$ to some function $u \in C^{2,1}(\mathbb{R}^n \times (0,\infty))$ as $k \to \infty$. Then u satisfies (0.11) in $\mathbb{R}^n \times (0,\infty)$.

Putting $u = u_k$ in (0.5) and letting $k \to \infty$ we get that u satisfies (0.5) in $\mathbb{R}^n \times (0,\infty)$. Since $G_k(x,y,t)$ increases monotonically to $(4\pi t)^{-n/2} e^{-|x-y|^2/4t}$ as $k \to \infty$, the first two terms on the right hand side of (1.21) converges to

$$\int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}} u_0(y)\, dy - \int_0^t \int_{\mathbb{R}^n} \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4(t-s)}} u(y,s)^{-\nu}\, dy\, ds$$

as $k \to \infty$. By Lemma 1.3 of [8] for any $T > 0$ there exists constants $C_1 > 0$, $c_1 > 0$ such that

$$\left| \frac{\partial G_k}{\partial N_y}(x,y,t-s) \right| \le C_1 \frac{k-|x|}{(t-s)^{\frac{n+2}{2}}} e^{-c_1 \frac{|x-y|^2}{t-s}} \quad \forall |x| < k, |y| = k, 0 < s < t < T, k \ge 1 \tag{1.22}$$

Let $k_0 > 1$. Then by (1.22) for any $|x| \le k_0$, $k \ge 2k_0$, $|y| = k$, $0 < s < t < T$,

$$\begin{aligned}
\left| \frac{\partial G_k}{\partial N_y}(x,y,t-s) \right| &\le C_1 \frac{k-|x|}{|x-y|^{n+2}} e^{-\frac{c_1|x-y|^2}{2(t-s)}} \left[\left(\frac{|x-y|^2}{(t-s)} \right)^{\frac{n+2}{2}} e^{-\frac{c_1|x-y|^2}{2(t-s)}} \right] \\
&\le C_2 \frac{k-|x|}{|x-y|^{n+2}} e^{-\frac{c_1|x-y|^2}{2(t-s)}} \\
&\le C_2 \frac{k-|x|}{(k-|x|)^{n+2}}
\end{aligned}$$

for some constant $C_2 > 0$. Hence

$$\left| \int_0^t \int_{|y|=k} \frac{\partial G_k}{\partial N_y}(x,y,t-s)(1+|y|^2+b_1 s)^{\alpha_1}\, d\sigma(y)\, ds \right| \le C \frac{(k-|x|)k^{n-1}}{(k-|x|)^{n+2}} (1+k^2)^{\alpha_1}$$
$$\to 0 \quad \text{as } k \to \infty \tag{1.23}$$

for any $|x| \le k_0, 0 < t < T$. Since $k_0 > 1$ and $T > 0$ are arbitrary, (1.23) holds for any $x \in \mathbb{R}^n$ and $t > 0$. Hence letting $k \to \infty$ in (1.21) we get (1.20).

By (0.5) the second term on the right hand side of (1.20) converges uniformly to 0 as $t \to 0$. Since the first term on the right hand side of (1.20) is the solution of the heat equation in $\mathbb{R}^n \times (0,\infty)$, this term converges in $L^1_{loc}(\mathbb{R}^n)$ to u_0 as $t \to 0$. Hence by (1.20) $u(\cdot, t)$ converges in $L^1_{loc}(\mathbb{R}^n)$ to u_0 as $t \to 0$. Thus u is a solution of (0.1). Suppose there exists another solution v of (0.1) which satisfies (0.5) and (1.20) in $\mathbb{R}^n \times (0,\infty)$. By the mean value theorem,

$$-u^{-\nu} + v^{-\nu} = \nu \xi^{-(1+\nu)}(u-v)$$

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for some $\xi = \xi(x,t)$ between $u(x,t)$ and $v(x,t)$. Then by (0.5),

$$\xi^{-(1+\nu)} \le C(1 + |x|^2 + b_1 t)^{-\alpha_1(1+\nu)} \le C \quad \forall x \in \mathbb{R}^n, t > 0,$$

for some constant $C > 0$. Henc by Lemma 1.4 $u = v$ on $\mathbb{R}^n \times (0, \infty)$. Thus the solution u of (0.1) is unique. $\qquad\square$

By Lemma 1.2, Lemma 1.4 and an argument similar to the proof of Theorem 1.5 we obtain the following extension of the local existence theorem of (0.1) (cf. Theorem 3.3 of [6]) proved in [6].

Theorem 1.6. *Let $\nu > 0$. Suppose $u_0 \in C(\mathbb{R}^n)$ satisfies $\delta = \inf_{\mathbb{R}^n} u_0 > 0$ and there exists constants $C_1 > 0$ and $C_2 > 0$ such that*

$$u_0(x) \le C_1 e^{C_2 |x|^2} \quad in \ \mathbb{R}^n.$$

Then there exists a constant $T > 0$ such that there exists a unique solution u of (0.1) in $\mathbb{R}^n \times (0,t)$ which satisfies (1.20) in $\mathbb{R}^n \times (0, \infty)$. If $u_0 \le C_3(1 + |x|^2)^\alpha$ in $\mathbb{R}^n$ for some constants $C_3 > 0$ and $\alpha \ge 1/(1+\nu)$, then $T \ge (\delta/2)^{-\nu-1}$ such that when $T < \infty$ we have $\lim_{t \nearrow T} \inf_{\mathbb{R}^n} u(\cdot, t) = 0$.

2 Extinction properties of solution

In this section we will establish various conditions on the initial value u_0 for the solutions of (0.1) to extinct in a finite time. We will give upper bound estimates for the extinction time and find the extinction rate of the solutions of (0.1).

Theorem 2.1. *Let $0 < \nu \le 1$,*

$$0 < \beta \le \frac{1}{1+\nu}.$$

Let $u_0 \in C(\mathbb{R}^n)$ satisfies

$$0 < u_0 \le A_3 T^{\frac{1}{1+\nu}} (1 + |x|^2)^{-\beta}$$

for some constant

$$A_3 = \begin{cases} (1+\nu)^{\frac{1}{1+\nu}} & \text{if } 0 < \beta \le \min((n-2)/2, 1/(1+\nu)) \\ \left(\dfrac{1+\nu}{1 + 2\beta(1+\nu)(2\beta + 2 - n)T} \right)^{\frac{1}{1+\nu}} & \text{if } (n-2)/2 < \beta \le 1/(1+\nu). \end{cases} \tag{2.1}$$

Suppose u is a solution of (0.1). Then

$$u(x,t) \le A_3(T-t)^{\frac{1}{1+\nu}} (1 + |x|^2)^{-\beta} \quad \forall x \in \mathbb{R}^n, 0 < t < T. \tag{2.2}$$

Proof: Suppose u is a solution of (0.1). Let $\psi_3(x,t) = A_3(T-t)^{\frac{1}{1+\nu}}(1+|x|^2)^{-\beta}$. Then $u_0 \le \psi_3(x,0)$ on $\mathbb{R}^n$. By direct computation,

$$\Delta\psi_3 = A_3(T-t)^{\frac{1}{1+\nu}}[2\beta(2\beta+2-n)(1+|x|^2)^{-\beta-1} - 4\beta(\beta+1)(1+|x|^2)^{-\beta-2}]$$

and

$$\psi_{3,t} = -\frac{A_3}{1+\nu}(T-t)^{-\frac{\nu}{1+\nu}}(1+|x|^2)^{-\beta}.$$

Hence

$$\Delta\psi_3 - \psi_3^{-\nu} - \psi_{3,t}$$

$$=A_3(T-t)^{\frac{1}{1+\nu}}(1+|x|^2)^{-\beta-2}\bigg(2\beta(2\beta+2-n)(1+|x|^2) - 4\beta(\beta+1)$$

$$- A_3^{-\nu-1}\frac{(1+|x|^2)^{2+\beta(1-\nu)}}{T-t} + \frac{1}{1+\nu}\cdot\frac{(1+|x|^2)^2}{T-t}\bigg)$$

$$\le A_3(T-t)^{\frac{1}{1+\nu}}(1+|x|^2)^{-\beta-2}\bigg(2\beta(2\beta+2-n)(1+|x|^2) - 4\beta(\beta+1)$$

$$+ \frac{(1+|x|^2)^2}{T-t}\cdot\bigg(\frac{1}{1+\nu} - A_3^{-\nu-1}\bigg)\bigg). \tag{2.3}$$

By (2.1) if $0 < \beta \le \min((n-2)/2, 1/(1+\nu))$, then the right hand side of (2.3) is ≤ 0. If $(n-2)/2 < \beta \le 1/(1+\nu)$, let

$$\varepsilon = \frac{1}{1 + 2\beta(1+\nu)(2\beta+2-n)T}.$$

Then by (2.1),

$$A_3 = (\varepsilon(1+\nu))^{\frac{1}{1+\nu}} = \bigg(\frac{1-\varepsilon}{2\beta(2\beta+2-n)T}\bigg)^{\frac{1}{1+\nu}}.$$

Hence

$$\frac{1}{1+\nu} = A_3^{-\nu-1}\varepsilon \tag{2.4}$$

and

$$2\beta(2\beta+2-n) = A_3^{-\nu-1}\frac{1-\varepsilon}{T}. \tag{2.5}$$

By (2.4) and (2.5) when $(n-2)/2 < \beta \le 1/(1+\nu)$, the right side of (2.3) is ≤ 0. Hence ψ_3 is a supersolution of (0.1). Then by Lemma 1.4,

$$u \le \psi_3 \quad \text{in } \mathbb{R}^n \times (0,T)$$

and (2.2) follows. $\qquad\square$

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Theorem 2.2. *Let $\nu > 0$, $0 < u_0 \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and $T = \|u_0\|_{L^\infty(\mathbb{R}^n)}^{1+\nu}/(1+\nu)$. Suppose u is a solution of (0.1). Then*

$$u(x,t) \le (1+\nu)^{\frac{1}{1+\nu}}(T-t)^{\frac{1}{1+\nu}} \quad \forall x \in \mathbb{R}^n, 0 < t < T. \tag{2.6}$$

Proof: Let $\psi_4(x,t) = (1+\nu)^{\frac{1}{1+\nu}}(T-t)^{\frac{1}{1+\nu}}$. Then $u_0 \le \psi_4(x,0)$ on $\mathbb{R}^n$ and

$$\Delta\psi_4 - \psi_4^{-\nu} - \psi_{4,t} = 0 \quad \text{in } \mathbb{R}^n \times (0,T).$$

Hence by Lemma 1.4,

$$u \le \psi_4 \quad \text{in } \mathbb{R}^n \times (0,T)$$

and (2.6) follows. $\qquad\square$

Theorem 2.3. *Let $\nu > 0$ and let $0 < u_0 \in C(\mathbb{R}^n)$ satisfies (0.8) for some constants $T_1 > 0$ and $A > 0$ satisfying (0.9). Let u be a solution of (0.1). Then there exist constants $T > 0$ and $b > 0$ such that*

$$u(x,t) \le A(b(T-t) + |x|^2)^{\frac{1}{1+\nu}} \quad \text{in } \mathbb{R}^n \times (0,T). \tag{2.7}$$

Hence u vanishes at $x = 0$ at time $t = T$.

Proof: Let

$$b = \frac{1+\nu}{A^{1+\nu}} - 2n \tag{2.8}$$

and $T = T_1/b$. By (0.9) $b > 0$. Let $\psi_5(x,t) = A(b(T-t) + |x|^2)^{\frac{1}{1+\nu}}$. Suppose u is a solution of (0.1). By (0.8)

$$u_0 \le \psi_5(x,0) \quad \text{on } \mathbb{R}^n. \tag{2.9}$$

By direct computation,

$$\Delta\psi_5 \le \frac{2nA}{1+\nu}(|x|^2 + b(T-t))^{-\frac{\nu}{1+\nu}} \quad \text{in } \mathbb{R}^n \times (0,T).$$

Hence by (2.8),

$$\begin{aligned}
&\Delta\psi_5 - \psi_5^{-\nu} - \psi_{5,t} \\
\le& \frac{2nA}{1+\nu}(|x|^2 + b(T-t))^{-\frac{\nu}{1+\nu}} - A^{-\nu}(|x|^2 + b(T-t))^{-\frac{\nu}{1+\nu}} + \frac{bA}{1+\nu}(|x|^2 + b(T-t))^{-\frac{\nu}{1+\nu}} \\
\le& 0 \quad \text{in } \mathbb{R}^n \times (0,T).
\end{aligned} \tag{2.10}$$

By (2.9), (2.10) and Lemma 1.4,

$$u \le \psi_5 \quad \text{on } \mathbb{R}^n \times (0,T)$$

and (2.8) follows. $\qquad\square$

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